# Screw instability in black hole magnetospheres and a stabilizing effect of field-line rotation

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The screw instability of magnetic field is a mechanism for prohibiting a generation of strongly twisted field lines in large scales. If it can work in black hole magnetospheres, the global axisymmetric structure and the main process of energy release will be significantly influenced. In this paper, we study the instability condition in the framework of the variational principle, paying special attention to a stabilizing effect due to field-line rotation against the screw-unstable modes satisfying the well-known Kruskal-Shafranov criterion. The basic formulation for the stability analysis of rotating force-free fields in Kerr geometry is provided. Then, for the practical use of the analytical method, the stationary configuration is assumed to be cylindrical, and we treat the long-wave mode perturbations excited at large distances from the central black hole, where the strength of gravity becomes negligibly small. This allows us to derive clearly the new criterion dependent on the rotational angular velocity of magnetic field lines, and the implications of the results obtained for magnetic activities of jets driven by a black hole are briefly discussed.

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## I. INTRODUCTION

The global structure of black hole magnetospheres involving axisymmetric magnetic field and plasma injected from an accretion disk has been extensively investigated for explaining various observational features of active galactic nuclei (AGNs) (see, e.g., [1] for a review). The magnetic field lines which are assumed to be frozen to the rotating plasma may be twisted and rotating, which means that the components of toroidal magnetic field (i.e., poloidal electric currents) and poloidal electric field are generated. The magnetic tension of the toroidal field can play an important role of forming cylindrically collimated jets over significantly long distances, and if the magnetic field lines thread a rotating black hole in the central region of AGN, the so-called Blandford-Znajek mechanism [2,3] is believed to become a possible process for powering the plasma jets through Poynting flux.

However, it is also well-known from the field of plasma physics that the magnetic field configurations with both poloidal component and toroidal one can be screw-unstable [4,5], and this process of a current driven instability makes the magnetic field lines untwist on short time-scales and causes a sudden release of magnetic free-energy to plasma's kinetic energy. In conventional plasma confinements, the screw instability is claimed to work for long-wave mode perturbations, if the equilibrium magnetic field has a toroidal component exceeding the limit given by the Kruskal-Shafranov criterion. Then, according to this instability condition one may discuss that the power of the Blandford-Znajek mechanism which is crucially dependent on the strength of of toroidal magnetic field is significantly suppressed [6], and the mechanism for acceleration and collimation of jets also becomes less efficient. Further, a flare-like energy radiation is expected to occur in the screw-unstable magnetospheres [7]. After a part of the magnetic energy is released through the process of the screw instability, the magnetic field lines threading a rotating black hole may be twisted again for generating the toroidal component stronger than the Kruskal-Shafranov threshold, and the flaring may be quasi-periodically observed.

Thus, the screw instability is an astrophysically interesting process which can significantly influence the global axisymmetric structure and the main process of energy release in black hole magnetospheres. For the application to the astrophysical problems, however, the instability condition should be investigated in more detail. In particular, an essential feature of black hole magnetospheres is that the twisted magnetic field lines are rotating. In this paper, our main purpose is to consider the effect of the field-line rotation in the stability analysis of stationary magnetic configurations and to derive the new criterion of the screw instability. In Sec. II, in the framework of the force-free approximation in Kerr geometry, we give the basic equations for the axisymmetric stationary fields and the non-axisymmetric linear perturbations, following the usual procedure to derive the variational principle [4,8]. For the practical use of the analytical method, in Sec. III, the stationary configuration is assumed to be cylindrical, and we

treat the long-wave mode perturbations excited at large distances from the central black hole, where the strength of gravity becomes negligibly small. Then we can clearly see that the field-line rotation has a stabilizing effect against the screw-unstable modes satisfying the Kruskal-Shafranov criterion. This allows us to reconsider the conclusions based on the Kruskal-Shafranov criterion as a first step toward more complete investigations using the fully general relativistic formulae given in Sec. II. While any direct influences of Kerr geometry are missed in the new criterion obtained here (which is relevant to force-free rotating magnetospheres far distant from various central objects), it is possible to discuss the implications for magnetic activities in black hole magnetospheres, by estimating the toroidal component of the stationary magnetic field under the assumption that it is generated as a result of the black hole rotation. Section IV is devoted to a brief comment on this problem. Throughout this paper, we use the geometric units with $G = c = 1$.

## II. STABILITY ANALYSIS IN KERR GEOMETRY

In the framework of the 3+1 formalism in Kerr geometry we study a current driven instability of electromagnetic fields under the force-free approximation which will be valid in black hole magnetospheres except near the event horizon and the (inner and outer) light cylinders [9]. The Kerr metric is written in the Boyer-Lindquist coordinates as follows,

$$ds^2 = -\alpha^2 dt^2 + h_{ij}(dx^j - \beta^j dt)(dx^i - \beta^i dt), \tag{2.1}$$

where

$$\alpha = \frac{\rho}{\Sigma}\sqrt{\Delta}, \quad \beta^r = \beta^\theta = 0, \quad \beta^\phi = \frac{2Mar}{\Sigma^2} \equiv \omega, \tag{2.2}$$

$$h_{rr} = \frac{\rho^2}{\Delta}, \quad h_{\theta\theta} = \rho^2, \quad h_{\phi\phi} = \varpi^2, \tag{2.3}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr, \tag{2.4}$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \quad \varpi = \frac{\Sigma}{\rho}\sin\theta, \tag{2.5}$$

and $M$ and $a$ denote the mass and angular momentum per unit mass of the black hole, respectively. Then, the Maxwell equations and the force-free condition for zero-angular-momentum observers (ZAMOs) are given by

$$(\partial_t + L_\beta)\vec{B} = -\nabla \times \alpha\vec{E}, \tag{2.6}$$

$$(\partial_t + L_\beta)\vec{E} = \nabla \times \alpha\vec{B} - \alpha\vec{j}, \tag{2.7}$$

$$\rho\vec{E} + \vec{j} \times \vec{B} = 0, \tag{2.8}$$

where $L_\beta$ denotes the Lie derivative along $\beta^i$, $\vec{B}$ and $\vec{E}$ are the magnetic and electric fields, $\rho$ and $\vec{j}$ are the electric charge and the current densities multiplied by $4\pi$ [2].

If the current and charge densities are eliminated from the Maxwell equations, we have the equation

$$\partial_t(\vec{B} \times \vec{E}) + \vec{B} \times L_\beta\vec{E} + (L_\beta\vec{B}) \times \vec{E} + \alpha(\vec{\nabla} \cdot \vec{E})\vec{E} = \vec{B} \times (\vec{\nabla} \times \alpha\vec{B}) + \vec{E} \times (\vec{\nabla} \times \alpha\vec{E}). \tag{2.9}$$

Because the force-free fields may be regarded as a magnetically dominated limit in magnetohydrodynamics (MHD), we introduce the plasma velocity defined by

$$\vec{E} + \vec{v} \times \vec{B} = 0, \tag{2.10}$$

and we have the frozen-in law written by

$$(\partial_t + L_\beta)\vec{B} = \vec{\nabla} \times (\vec{v} \times \alpha\vec{B}). \tag{2.11}$$

These are the basic set of dynamical equations for the force-free fields.

For axisymmetric stationary configurations of electromagnetic and velocity fields denoted by $\vec{B}_0$, $\vec{E}_0$ and $\vec{v}_0$, we obtain the well-known relations. For example, the poloidal component $\vec{B}_0^p$ of the magnetic field is written by the magnetic stream function $\Psi$ as follows,

$$\vec{B}_0^p = \frac{\nabla\Psi \times \vec{e}_T}{2\pi\varpi}, \tag{2.12}$$

where $\vec{e}_T = (1/\varpi)(\partial/\partial\phi)$ means the toroidal basis vector which has unit norm. In stationary rotating magnetospheres the velocity field is given by

$$\vec{v}_0 \equiv v_0^T \vec{e}_T = (\varpi/\alpha)(\Omega_F - \omega)\vec{e}_T, \tag{2.13}$$

where $\Omega_F = \Omega_F(\Psi)$ is constant along a magnetic field line of $\Psi(r,\theta) =$ constant, and we call it the angular velocity of a magnetic field line. This rotational motion generates the poloidal electric field $\vec{E}_0 = -\vec{v}_0 \times \vec{B}_0$. Further, if the magnetic field lines are twisted, the poloidal current $I = I(\Psi)$ (which is chosen to be positive for downward flows) is generated according to the equation

$$I = -\alpha\varpi B_0^T/2, \tag{2.14}$$

where $B_0^T$ is the toroidal magnetic component. Then, Eq. (2.9) reduces to the Grad-Shafranov equation for $\Psi$ [10],

$$\frac{\Omega_F - \omega}{\alpha}(\vec{\nabla}\Psi)^2 \frac{d\Omega_F}{d\Psi} + \vec{\nabla}\cdot\left\{\frac{\alpha}{\varpi^2}\left[1 - \left(\frac{\Omega_F - \omega}{\alpha}\varpi\right)^2\right]\vec{\nabla}\Psi\right\} + \frac{16\pi^2}{\alpha\varpi^2} I \frac{dI}{d\Psi} = 0, \tag{2.15}$$

which should be solved by giving the functions $I(\Psi)$ and $\Omega_F(\Psi)$ from boundary conditions at a plasma source. Though we do not try to solve the highly nonlinear equation (2.15) for background fields, it will be used for simplifying the equations for linear perturbations in the stability analysis in the following.

Now, we consider small perturbations $\vec{B}_1$, $\vec{E}_1$ and $\vec{v}_1$ from the stationary configurations. It is easy to derive the linearized versions of Eqs. (2.9), (2.10) and (2.11) as follows,

$$\vec{E}_1 = -\vec{v}_0 \times \vec{B}_1 - \vec{v}_1 \times \vec{B}_0, \tag{2.16}$$

$$(\partial_t + L_\beta)\vec{B}_1 = \vec{\nabla} \times (\vec{v}_1 \times \alpha\vec{B}_0 + \vec{v}_0 \times \alpha\vec{B}_1), \tag{2.17}$$

$$\begin{aligned} \vec{B}_0 \times \partial_t \vec{E}_1 + \vec{B}_0 \times L_\beta \vec{E}_1 + \vec{B}_1 \times L_\beta \vec{E}_0 + \alpha\left[(\nabla\cdot\vec{E}_0)\vec{E}_1 + (\nabla\cdot\vec{E}_1)\vec{E}_0\right] \\ = \vec{B}_0 \times (\vec{\nabla}\times\alpha\vec{B}_1) + \vec{B}_1 \times (\vec{\nabla}\times\alpha\vec{B}_0). \end{aligned} \tag{2.18}$$

Because the background fields and geometry are stationary and axisymmetric, the perturbed fields can have the form of $\exp(im\phi - i\sigma t)$, where $m$ is an arbitrary integer and $\sigma$ is an arbitrary complex number. The key point for analyzing the linear equations is to introduce the plasma displacement $\vec{\xi}$ defined by $\vec{v}_1 = -i\sigma\vec{\xi}$, for which we can give $\vec{\zeta} \equiv \vec{\xi} \times \vec{B}_0$. Then, from Eqs. (2.16) and (2.17), the perturbed fields $\vec{B}_1$ and $\vec{E}_1$ are explicitly solved as follows,

$$\vec{B}_1 = \nu(\vec{\nabla}\times\alpha\vec{\zeta}\,) + i\frac{(\vec{B}_1^p\cdot\vec{\nabla}\Omega_F)}{\sigma - m\Omega_F}\varpi\vec{e}_T, \tag{2.19}$$

$$\vec{E}_1 = \nu\frac{\varpi(\Omega_F - \omega)}{\alpha}\left[(\vec{\nabla}\times\alpha\vec{\zeta})\times\vec{e}_T\right] + i\sigma\vec{\zeta}, \tag{2.20}$$

where $\nu \equiv \sigma/(\sigma - m\Omega_F)$.

For mathematical simplicity in later calculations, we assume that $\Omega_F$ is constant all over the field lines, i.e., the second term of the right-hand side of Eq. (2.19) vanishes. Then, by substituting $\vec{B}_1$ and $\vec{E}_1$ into Eq. (2.18), our problem reduces to the eigenvalue problem for the eigenfrequency $\sigma$ and the eigenvector $\vec{\zeta}$ (or $\vec{\xi}$ ), which has the form

$$\sum_{n=0}^{2} \sigma^n(\hat{A}_n\vec{\xi}\,) = 0. \tag{2.21}$$

Because we use the variational principle to consider the eigenvalue problem, here we express the differential operators $\hat{A}_n$ in terms of the integrals of arbitrary displacement vectors $\vec{\xi}$ and $\vec{\eta}$ as follows,

$$a_n \equiv \int d^3r\,\sqrt{h}\;\vec{\eta}^{\,*}\cdot(\hat{A}_n\vec{\xi}\,) \equiv \int d^3r\;\alpha\sqrt{h}\;b_n, \tag{2.22}$$

where stars and $h$ denote complex conjugates and the determinant of the metric $h_{ij}$, respectively. Some parts of the integrands $\vec{\eta}^{\,*}\cdot(\hat{A}_n\vec{\xi}\,)$ reduce to the total divergence written by $\vec{\nabla}\cdot\vec{b}$, which is assumed to vanish owing to the

condition such that no perturbations are generated at the boundaries. Then, through the lengthy calculations using $\vec{\zeta}$ and $\vec{\chi} \equiv \vec{\eta} \times \vec{B}_0$ instead of $\vec{\xi}$ and $\vec{\eta}$, we arrive at the results

$$b_2 = \vec{\chi}^{\,*} \cdot \vec{\zeta}, \tag{2.23}$$

$$\begin{aligned} b_1 = & -2m\omega\vec{\chi}^{\,*} \cdot \vec{\zeta} + i\frac{\alpha(\vec{\nabla} \cdot \vec{E}_0)}{B_0^2}\vec{B}_0 \cdot (\vec{\chi}^{\,*} \times \vec{\zeta}\,) \\ & + \frac{i}{\alpha}(\Omega_F - \omega)\left[\vec{\chi}^{\,*} \cdot \vec{\nabla}(\varpi\alpha\zeta_T) - \vec{\zeta} \cdot \vec{\nabla}(\varpi\alpha\chi_T^*)\right], \end{aligned} \tag{2.24}$$

$$\begin{aligned} b_0 = & \, m^2\omega^2\vec{\chi}^{\,*} \cdot \vec{\zeta} + \frac{(\Omega_F - \omega)^2}{\alpha^2}\vec{\nabla}(\varpi\alpha\chi_T^*) \cdot \vec{\nabla}(\varpi\alpha\zeta_T) \\ & - i\frac{m\alpha\Omega_F(\vec{\nabla} \cdot \vec{E}_0)}{B_0^2}\vec{B}_0 \cdot (\vec{\chi}^{\,*} \times \vec{\zeta}\,) - \frac{im\omega}{\alpha}(\Omega_F - \omega)\left[\vec{\chi}^{\,*} \cdot \vec{\nabla}(\varpi\alpha\zeta_T) - \vec{\zeta} \cdot \vec{\nabla}(\varpi\alpha\chi_T^*)\right] \\ & - (\vec{\nabla} \times \alpha\vec{\chi}^{\,*}) \cdot (\vec{\nabla} \times \alpha\vec{\zeta}\,) - 2\pi\left(\frac{dI}{d\Psi}\right)\left[\vec{\chi}^{\,*} \cdot (\vec{\nabla} \times \alpha\vec{\zeta}\,) + \vec{\zeta} \cdot (\vec{\nabla} \times \alpha\vec{\chi}^{\,*})\right] \end{aligned} \tag{2.25}$$

where $\zeta_T$ and $\chi_T$ are the toroidal components of $\vec{\zeta}$ and $\vec{\chi}$ measured in the orthonormal bases. These equations clearly show that the operators $\hat{A}_n$ are Hermitian.

If the vector $\vec{\chi}$ is chosen to be equal to $\vec{\zeta}$, the coefficients $a_n$ appearing in the dispersion relation

$$a_2\sigma^2 + a_1\sigma + a_0 = 0 \tag{2.26}$$

become real numbers. Because $a_2$ is positive definite, it is clear that the sufficient and necessary condition for the screw instability of stationary configurations is

$$W \equiv W_P + W_R < 0, \tag{2.27}$$

where $W_P \equiv -a_0$ is the so-called potential energy for the small displacement $\vec{\xi}$ [4,8]. Even if $W_P$ is not positive semi-definite, stationary configurations in rotating magnetospheres can be stabilized by virtue of the existence of the rotational term $W_R \equiv a_1^2/4a_2$.

It should be remarked that $a_n$ are the functional of $\vec{\zeta}$ which is perpendicular to $\vec{B}_0$. This motivates us to introduce the poloidal orthonormal basis vectors defined by

$$\vec{e}_{\parallel} = \frac{\vec{B}_0^p}{|B_0^p|}, \quad \vec{e}_{\perp} = \frac{\vec{\nabla}\Psi}{2\pi\varpi|B_0^p|}, \tag{2.28}$$

in addition to the toroidal basis vector $\vec{e}_T$ with the condition $\vec{e}_{\parallel} = \vec{e}_{\perp} \times \vec{e}_T$. Then, decomposing the vector $\vec{\zeta}$ into

$$\vec{\zeta} = \zeta_{\parallel}\vec{e}_{\parallel} + \zeta_{\perp}\vec{e}_{\perp} + \zeta_T\vec{e}_T, \tag{2.29}$$

we obtain

$$\zeta_T = -\xi_{\perp}B_0^p, \quad \zeta_{\parallel} = \xi_{\perp}B_0^T \tag{2.30}$$

for $\vec{B}_0 = B_0^p\vec{e}_{\parallel} + B_0^T\vec{e}_T$ and $\xi_{\perp} = \vec{e}_{\perp} \cdot \vec{\xi}$. Hence, we can treat the two functions $\xi_{\perp}$ and $\zeta_{\perp}$ as the trial functions to calculate the eigenvalue $\sigma$.

We have established the basic formulae to use the variational principle for the stability analysis in Kerr geometry. The next step will be to apply the method to Eq. (2.26) with the purpose of determining the trial functions and obtaining the instability criterion expressed by stationary background fields. Unfortunately, this becomes a very complicated task owing to the geometrical terms involved in $a_n$. Then, our interest in the next section is focused on a stabilizing effect of $\Omega_F$ in rotating magnetospheres, considering simple stationary fields and the perturbations given in flat spacetime.

## III. STABILIZING EFFECT IN ROTATING MAGNETOSPHERES

As a mechanism of jet formation the existence of collimated field lines threading a black hole is astrophysically interesting. The jets would propagate along the poloidal magnetic field lines to large distances, keeping the collimated

structure under an action of the toroidal magnetic field. Here, the stationary magnetic field $\vec{B}_0$ is assumed to be cylindrical as a typical configuration of collimated field lines. We discuss the instability of such a global structure due to the perturbations $\vec{\xi}$ with a wave length (in the vertical direction along the filed line) much longer than the horizon scale. (Long-wave modes are expected to become dominant in the screw instability [4].) Then, the integrals in the vertical direction appearing in $a_n$ should extend to a very large scale such that $L \gg M$, where gravity becomes very weak. This allows us to estimate approximately the integrals $a_n$ by giving fields in flat spacetime.

Because of the cylindrical configuration of the stationary magnetic field, it is convenient to use the cylindrical coordinates denoted by $\varpi$, $\phi$ and $z$, for which the Euclidean metric $h_{ij}$ is given by

$$h_{ij}dx^i dx^j = d\varpi^2 + \varpi^2 d\phi^2 + dz^2, \tag{3.1}$$

and the orthonormal basis vectors previously introduced become

$$\vec{e}_{\parallel} = \vec{e}_z, \quad \vec{e}_{\perp} = \vec{e}_{\varpi}. \tag{3.2}$$

The magnetic components are assumed to be dependent on $\varpi$ only as follows,

$$\vec{B}_0 = B(\varpi)\vec{e}_z + U(\varpi)\vec{e}_T, \tag{3.3}$$

and for $\alpha = 1$ and $\omega = 0$ the Grad-Shafranov equation (2.15) reduces to

$$(1 - \varpi^2\Omega_F^2)BB' + UU' = 2\varpi\Omega_F^2 B^2 - \frac{1}{\varpi}U^2, \tag{3.4}$$

where primes denote the differentiation with respect to $\varpi$. To derive Eq. (3.4), we have also used the relation

$$\frac{dI}{d\Psi} = -\frac{(\varpi U)'}{4\pi\varpi B} \tag{3.5}$$

for the poloidal current $I$. If the toroidal component $U$ is given, we can determine the distribution of the poloidal component $B$. The simplest example giving a constant $B$ would be $U = \pm\varpi\Omega_F B$.

Now the perturbations with the eigenfrequency $\sigma$ can have the form

$$\begin{aligned} \xi_{\perp} = \xi_{\varpi} = \xi(\varpi)\ \exp(im\phi + ikz - i\sigma t), \\ \zeta_{\perp} = \zeta_{\varpi} = \zeta(\varpi)\ \exp(im\phi + ikz - i\sigma t), \end{aligned} \tag{3.6}$$

where $k$ is the wave number in the $z$-direction, and the minimal value of $k$ becomes approximately $L^{-1}$ in the magnetosphere with the scale $L$. To obtain a relation between the two trial functions $\xi$, $\zeta$, we consider the variation of Eq. (2.26). As will be seen later, the variational principle becomes consistent if $\vec{\eta}$ is chosen to be an eigenvector with the eigenvalue $\sigma^*$ (which may be complex), i.e.,

$$\begin{aligned} \eta_{\perp} = \eta_{\varpi} = \eta(\varpi)\ \exp(im\phi + ikz - i\sigma^* t), \\ \chi_{\perp} = \chi_{\varpi} = \chi(\varpi)\ \exp(im\phi + ikz - i\sigma^* t), \end{aligned} \tag{3.7}$$

where $\eta = \xi(\varpi, \sigma^*)$ and $\chi = \zeta(\varpi, \sigma^*)$.

Using these eigenvectors $\vec{\zeta}$ and $\vec{\chi}$, the coefficients $a_n$ in Eq. (2.26) reduce to

$$a_n = 2\pi L \int \varpi d\varpi\ b_n. \tag{3.8}$$

with $L$ as the length in the $z$-direction. It is easy to calculate the integrands $b_2$ and $b_1$ with the results

$$b_2 = \chi^*\zeta + (B^2 + U^2)\eta^*\xi, \tag{3.9}$$

$$b_1 = i\Omega_F\left[\zeta(\varpi B\eta'^* - B\eta^*) - \chi^*(\varpi B\xi' - B\xi) + 2i(mB - k\varpi U)B\eta^*\xi\right]. \tag{3.10}$$

To calculate $b_0$, the partial integration of the terms containing $(\eta^*\xi)'$ becomes necessary. By omitting the contribution from the boundary, we have

$$b_0 = P\eta'^*\xi' + Q\eta^*\xi + \frac{m^2 + k^2\varpi^2}{\varpi^2}\left(i\chi^* - \frac{S_\eta^*}{m^2 + k^2\varpi^2}\right)\left(i\zeta + \frac{S_\xi}{m^2 + k^2\varpi^2}\right), \tag{3.11}$$

where

$$P = -\frac{(mU + k\varpi B)^2}{m^2 + k^2\varpi^2} + (\varpi\Omega_F B)^2, \tag{3.12}$$

$$Q = \frac{(mU + k\varpi B)^2}{\varpi^2}\left(\frac{1}{m^2 + k^2\varpi^2} - 1\right) - \frac{2k^2(k^2\varpi^2B^2 - m^2U^2)}{(m^2 + k^2\varpi^2)^2}$$
$$- \Omega_F^2\left[(1 - m^2 - k^2\varpi^2)B^2 - \frac{2k^2\varpi^2(\varpi BB' + 2B^2)}{m^2 + k^2\varpi^2}\right], \tag{3.13}$$

$$S_\xi = (mB - k\varpi U)\varpi\xi' + (mB + k\varpi U)\xi, \tag{3.14}$$

$$S_\eta = (mB - k\varpi U)\varpi\eta' + (mB + k\varpi U)\eta. \tag{3.15}$$

It should be noted that no derivatives of $\zeta$ and $\chi$ appears in the integrands. Then, taking the variation $\delta$ of Eq. (2.26) and putting $\delta\xi = \delta\eta^* = 0$, we require that the coefficients of $\delta\zeta$ and $\delta\chi^*$ vanish, because they are algebraically independent and arbitrary. This procedure leads to the results

$$(m^2 + k^2\varpi^2 - \sigma^2\varpi^2)\zeta = iS_\xi - i\Omega\sigma\varpi^2(\varpi B\xi' - B\xi), \tag{3.16}$$

and

$$(m^2 + k^2\varpi^2 - \sigma^2\varpi^2)\chi^* = -iS_\eta^* + i\Omega\sigma\varpi^2(\varpi B\eta'^* - B\eta^*), \tag{3.17}$$

while the latter is automatically satisfied by virtue of $\eta = \xi(\varpi, \sigma^*)$ and $\chi = \zeta(\varpi, \sigma^*)$.

Since $\zeta$ has been fixed by the variational method, the dispersion relation (2.26) is now given by

$$\int \varpi d\varpi(F\eta'^*\xi' + G\eta^*\xi + H_\sigma) = 0, \tag{3.18}$$

where

$$F = -\frac{(mU + k\varpi B)^2}{m^2 + k^2\varpi^2}, \tag{3.19}$$

$$G = \frac{(mU + k\varpi B)^2}{\varpi^2}\left(\frac{1}{m^2 + k^2\varpi^2} - 1\right) - \frac{2k^2(k^2\varpi^2B^2 - m^2U^2)}{(m^2 + k^2\varpi^2)^2} - \frac{m^2\Omega_F^2(\varpi^4B^2)'}{\varpi^3(m^2 + k^2\varpi^2)}, \tag{3.20}$$

and the term dependent on $\sigma$ is

$$H_\sigma = \frac{m^2 + k^2\varpi^2}{m^2 + k^2\varpi^2 - \sigma^2\varpi^2}$$
$$\times \left[\frac{\sigma S_\eta^*}{m^2 + k^2\varpi^2} - \Omega_F B(\varpi\eta'^* - \eta^*)\right]\left[\frac{\sigma S_\xi}{m^2 + k^2\varpi^2} - \Omega_F B(\varpi\xi' - \xi)\right]$$
$$+ \left[(\sigma - m\Omega_F)^2B^2 + (\sigma U + k\varpi\Omega_F B)^2\right]\eta^*\xi. \tag{3.21}$$

If $\sigma$ is real, the term $F\eta'^*\xi'$ is negative semi-definite. Then, if it becomes dominant in Eq. (3.18), the third term $H_\sigma$ can be positive for giving consistently a real eigenvalue. This means that the most dangerous mode to make $\sigma$ complex is obtained by choosing $\xi$ in such a way that the integral involving the term $F\eta'^*\xi'$ vanishes. In fact, for $\Omega_F = 0$ [7], the screw instability has been found to occur for the mode given by

$$\xi = \begin{cases} 1\ , & \text{for}\ \ \varpi \le \varpi_0, \\ \{\varpi_0(1+\mu) - \varpi\}/(\varpi_0\mu)\ , & \text{for}\ \ \varpi_0 \le \varpi \le \varpi_0(1+\mu), \\ 0\ , & \text{for}\ \ \varpi_0(1+\mu) \le \varpi, \end{cases} \tag{3.22}$$

where $\varpi_0$ is the first zero of $mU + k\varpi B$ which is assumed to be present in the range $0 < \varpi < \infty$. Taking the limit $\mu \to 0$, we can easily estimate the integral of $F\eta'^*\xi'$ to be of the order of $\mu$. (This trial function is defined without including $\sigma$, and so we have $\xi = \eta$.) We use the same trial function to analyze the dispersion relation (3.18) extended to the case of $\Omega_F \neq 0$.

Let us assume that $B$ is everywhere positive. For regularity of the toroidal field on the symmetry axis, $U$ should approach zero at least in proportion to $\varpi$ in the limit $\varpi \to 0$. Further let the wave number $k$ be positive. Then, the

sign of $m$ is chosen according to the requirement such that $mU < 0$. By virtue of the steep gradient of $\xi$ at $\varpi = \varpi_0$, the integral of $H_\sigma$ containing the derivatives $\eta'^*\xi'$ becomes of the order of $1/\mu$ in contradiction to Eq. (3.18), unless the eigenvalue $\sigma$ is suitably determined. It is easy to see that such a divergent term vanishes, if $\sigma$ is given by

$$\sigma = m\Omega_F + \sigma_1. \tag{3.23}$$

The small correction $\sigma_1$ turns out to be of the order of $\mu^{1/2}$, because in the limit $\mu \to 0$ Eq. (3.18) reduces to

$$\frac{m^2 + k^2\varpi_0^2}{m^2(1-\Omega_F^2\varpi_0^2) + k^2\varpi_0^2}\frac{\varpi_0^2 B_0^2}{m^2}\frac{\sigma_1^2}{\mu} = \int_0^{\varpi_0} \frac{d\varpi}{\varpi} J, \tag{3.24}$$

where $B_0 = B(\varpi_0)$, and we have

$$J = (mU + k\varpi B)^2 \left[1 - \Omega_F^2\varpi^2 - \frac{1+\Omega_F^2\varpi^2}{m^2(1-\Omega_F^2\varpi^2)+k^2\varpi^2}\right] + (m^2U^2 - k^2\varpi^2B^2)\frac{2(m^2\Omega_F^2 - k^2)\varpi^2}{[m^2(1-\Omega_F^2\varpi^2)+k^2\varpi^2]^2}. \tag{3.25}$$

Though the eigenvalue of $\sigma_1$ is not determined unless the gradient $\mu^{-1}$ of $\xi$ at $\varpi = \varpi_0$ is precisely given, it is easy to see the sign of $\sigma_1^2$ from Eq. (3.24). We note that the stability of stationary fields may drastically change if the light cylinder (i.e., $\Omega_F\varpi = 1$) is present in the range $0 < \varpi < \varpi_0$. However, our simple force-free model would not remain approximately valid in a region beyond the light cylinder where plasma inertia becomes very important. Hence, we would like to limit our stability analysis to the fields inside the light cylinder, i.e., $\varpi_0\Omega_F < 1$, and we can conclude that the stationary fields given by $B$ and $U$ become unstable if the condition

$$\int_0^{\varpi_0} \frac{d\varpi}{\varpi} J < 0 \tag{3.26}$$

is satisfied. This is equivalent to Eq. (2.27), if $W$ is estimated by using the eigenfrequency $\sigma = \Omega_F$ and the trial functions given by Eqs. (3.16) and (3.22).

From the first term proportional to $(mU + k\varpi B)^2$ in $J$ the screw mode with $|m| = 1$ (and $mU = -|U|$) turns out to be most important for the instability. To compare clearly the instability condition with the usual Kruskal-Shafranov criterion, let both $k$ and $\Omega_F$ be positive and small, i.e., $k\varpi_0 \ll 1$ and $\Omega_F\varpi_0 \ll 1$. For the long-wave screw mode giving $mU = -|U|$ in slowly rotating magnetospheres we have

$$J = -(|U| - k\varpi B)(k^2 + \Omega_F^2)\varpi^2 \left[|U| - k\varpi B + \frac{4(k^2 - \Omega_F^2)}{k^2 + \Omega_F^2} k\varpi B\right]. \tag{3.27}$$

Then, if $\Omega_F < k$, it is clear that the requirement $|U| > k\varpi B$ in the range $0 < \varpi < \varpi_0$ makes $J$ negative. Because the minimal value of the wave number $k$ is of the order of $L^{-1}$, we arrive at the Kruskal-Shafranov criterion

$$\frac{|U|}{\varpi} > \frac{B}{L} \tag{3.28}$$

as the instability condition for a twisted magnetic tube with the width $\varpi$. The important point obtained here is that the minimal value of $k$ is also restricted by $\Omega_F$, and we can only claim that the screw instability surely occurs if the condition

$$\frac{|U|}{\varpi} > \Omega_F B \tag{3.29}$$

is also satisfied.

If the condition (3.29) breaks down, the stability becomes a more subtle problem depending on the distribution of $B$ and $U$ as functions of $\varpi$. For example, let us consider a stationary model such that the poloidal field is nearly uniform ($B \simeq B_0$), while the weak toroidal field (i.e., $|U| \ll B$) is given by $|U| = \Omega_0\varpi B_0$ in the range $0 < \varpi < \varpi_1$ and $|U| = \Omega_0\varpi_1 B_0$ in the range $\varpi_1 < \varpi$. Though the weak dependence of $B$ on $\varpi$ may be calculated from Eq. (3.4), it is not important when we estimate the integral of $J$. The first zero of $|U| - k\varpi B$ appears at $\varpi = \varpi_0 = \Omega_0\varpi_1/k$, and the condition $|U| > k\varpi B$ in the range $0 < \varpi < \varpi_0$ leads to $k < |\Omega_0|$. Then, through the calculation of the integral of $J$, we can easily show that the stationary model with $|\Omega_0| \le \Omega_F$ becomes stable even for the screw mode satisfying the Kruskal-Shafranov criterion.

However, one may consider a different model with the nearly uniform poloidal field and the weak toroidal field given by $|U| = \Omega_0 \varpi B_0 \left[1 - (\varpi/\varpi_2)^2\right]^{1/2}$ in the range $0 < \varpi < \varpi_2$ and $|U| = 0$ in the range $\varpi_2 < \varpi$. Then, if the first zero $\varpi_0$ is chosen to be very close to $\varpi_2$, we have $|U(\varpi)| \gg |U(\varpi_0)| = k\varpi_0 B > k\varpi B$ (i.e., $J \simeq -U^2\Omega_F^2\varpi^2 < 0$) in the range $0 < \varpi < \varpi_0$, except near $\varpi = \varpi_0$. The stabilizing effect of $\Omega_F$ becomes very weak for the stationary model with $|U|$ which rapidly decreases as $\varpi$ increases.

The conclusion that the latter model with a decreasing $|U|$ is screw-unstable will be premature when the toroidal field vanishes at $\varpi = \varpi_2$ very close to the light cylinder (i.e., $\Omega_F\varpi_2 \simeq 1$). This is because we must use Eq. (3.25) for $J$, when $k$ is chosen to be very small according to the requirement $\varpi_0 \simeq \varpi_2$. Now we obtain the approximate form

$$J \simeq -U^2\Omega_F^2\varpi^2 \frac{1 - 4\Omega_F^2\varpi^2 + \Omega_F^4\varpi^4}{(1 - \Omega_F^2\varpi^2)^2}, \tag{3.30}$$

except near $\varpi = \varpi_0$, and it is easy to see that the integral of $J$ becomes positive by virtue of $\Omega_F\varpi_0 \simeq 1$. Thus, we can expect that the rotating field is well stabilized against the screw mode with $k$ smaller than $\Omega_F$, unless a significant decrease of the toroidal field occurs in an inner region distant from the light cylinder.

In summary, we have presented the new criterion (3.26), in which $J$ is given by Eq.(3.27) for long-wave screw modes in slowly rotating magnetospheres. If the more precise form (3.25) of $J$ is used, we can show a strong stabilizing effect due to the existence of the light cylinder. It is sure that the Kruskal-Shafranov criterion (3.28) is not a sufficient condition for the screw instability if the twisted magnetic tube is rotating. Importantly, in rotating magnetospheres with $\Omega_F > L^{-1}$, the toroidal magnetic field may be stably amplified to the maximum value given by $|U| = \Omega_F\varpi B$, instead of $|U| = \varpi B/L$. So, we can say that the stationary magnetic field (3.3) is quite a reasonable assumption as a global field structure for the jets except local structures as a knot. If the deviation from the cylindrical field is serious, the criterion based on the integral (3.26) would not be simply applicable. However, by rough estimations, we can expect that the stability effect by $\Omega_F$ remains valid in modified criterion. This will be crucial for discussing highly energetic phenomena in rotating magnetospheres.

## IV. DISCUSSION

Now we would like to give a brief comment on the application of the results obtained in the previous section to black hole magnetospheres. Let us assume that radial magnetic field lines thread a Kerr black hole which is slowly rotating with the angular velocity $\omega_H$. The split monopole magnetic field [3] is given by

$$\Psi = -2\pi c_0 \cos\theta, \quad B_0^T = c_0(\Omega_F - \omega_H) \sin^2\theta \tag{4.1}$$

with a constant $c_0$, from which we have

$$B_0^T \simeq (\Omega_F - \omega_H)\varpi B_0^p \tag{4.2}$$

in the region distant from the event horizon. As $r$ increases, the radial magnetic field lines would tend to be collimated toward a cylindrical configuration, conserving the magnetic flux $\varpi^2 B_0^p$ and the electric current $\varpi B_0^T$. Then, the stationary cylindrical field, for which the instability condition was discussed, would satisfy the relation

$$U \simeq (\Omega_F - \omega_H)\varpi B, \tag{4.3}$$

for the field lines threading the central black hole, i.e., in a limited range such that $0 < \varpi < \varpi_1$.

The constraint on the angular velocity $\Omega_F$ of the magnetic field lines has been studied in [6] using the Kruskal-Shafranov criterion. However, if the length $L$ of the magnetic tube is larger than $\Omega_F^{-1}$, the sufficient condition of the screw instability should be replaced by Eq. (3.29), which leads to

$$|\Omega_F - \omega_H| > \Omega_F, \tag{4.4}$$

and we obtain

$$\Omega_F < \omega_H/2. \tag{4.5}$$

If the magnetic field rotates too slowly (see, e.g., [11] for such a black-hole driven wind model), the toroidal field generated by the black hole becomes too strong to be stably sustained against the screw modes. This may yield a flaring event in plasma jets, because the stored magnetic energy is released to the kinetic energy of surrounding plasma [7].

If the magnetic field lines threading the black hole satisfy the condition

$$\Omega_F \geq \omega_H/2, \tag{4.6}$$

a stable configuration becomes possible. In particular, the Blandford-Znajek mechanism [3] can work as a steady process of energy extraction from the rotating black hole, if $\Omega_F$ is in the range $\omega_H/2 \leq \Omega_F < \omega_H$. Interestingly, the magnetic field can be marginally stable, when the power becomes maximum, i.e., $\Omega_F = \omega_H/2$. Thus, the jets can be significantly powered by the Blandford-Znajek mechanism at large distances $L$ far from the central black hole, contrary to the claim in [6].

The force-free model (4.1) is based on the boundary condition at the event horizon, which may not be justified if the effect of plasma accretion is taken into account. If we relax the constraint at the event horizon, the relation (4.3) would not always hold. Though the condition $\Omega_F = \omega_H/2$ of the marginal stability could be modified, it is plausible that the ratio $\omega_H/\Omega_F$ is a crucial parameter for determining a dominant energy source which powers jets produced in black hole magnetospheres: The screw instability will yield a helical structure of plasma jets and flare-like emission through the processes such as shock waves, while the Blandford-Znajek mechanism will provide steady Poynting flux for plasma acceleration. The physical mechanism to determine uniquely the parameter $\omega_H/\Omega_F$ is also controversial [12]. If various values of the ratio $\omega_H/\Omega_F$ are permissible for rotating black hole magnetospheres in AGNs, such two different types of jets may be observable.

There exists also the Newtonian solution of Eq. (2.15) for a rotating, paraboloidal and disk-connected field [13], satisfying the condition $B_0^T = -\Omega_F \varpi B_0^p$ at large distances. If the same instability condition holds even for the paraboloidal field, such a stationary model would be marginally stable. In this paper, we have considered magnetic field lines connecting a central black hole with remote astrophysical loads, in relation to jet formation. As was suggested in [7], closed field lines connecting the black hole with a surrounding disk may also become screw-unstable (see [14] for a non-rotating model). Thus, the analysis of the screw instability of various stationary fields including the effect of strong gravity is a necessary task for understanding its astrophysical significance in rotating black hole magnetospheres, and the basic formulae discussed in Sec.II for the stability analysis in Kerr geometry should be fully applied. The results of this paper would be very useful as the foundation of the future problems.

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